

AN ANALOGUE OF A THEOREM OF KURZWEIL

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ABSTRACT. A theorem of Kurzweil ('55) on inhomogeneous Diophantine approximation states that if θ is an irrational number, then the following are equivalent: (A) for every decreasing positive function ψ such that $\sum_{q=1}^{\infty} \psi(q) = \infty$, and for almost every $s \in \mathbb{R}$, there exist infinitely many $q \in \mathbb{N}$ such that $\|q\theta - s\| < \psi(q)$, and (B) θ is badly approximable. This theorem is not true if one adds to condition (A) the hypothesis that the function $q \mapsto q\psi(q)$ is decreasing. In this paper we find a condition on the continued fraction expansion of θ which is equivalent to the modified version of condition (A). This expands on a recent paper of D. H. Kim ('14).

An irrational number θ is said to be *badly approximable* (or *of bounded type*) if there exists $\varepsilon > 0$ such that for every rational $p/q \in \mathbb{Q}$,

$$\left| \theta - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^2}.$$

It is well-known that an irrational number θ is badly approximable if and only if the partial quotients of θ form a bounded sequence. Another equivalent condition was given by Kurzweil [6]. To state it, let us define the set

$$W(\theta, \psi) = \{s \in \mathbb{R} : \exists^\infty q \in \mathbb{N} \ \|q\theta - s\| < \psi(q)\},$$

where $\|\cdot\|$ denotes distance to the nearest integer. Then Kurzweil's result may be stated as follows: θ is badly approximable if and only if for every decreasing function $\psi : \mathbb{N} \rightarrow (0, \infty)$ such that $\sum_{q=1}^{\infty} \psi(q) = \infty$, the set $W(\theta, \psi)$ has full measure. (Note that if $\sum_{q=1}^{\infty} \psi(q) < \infty$, then the set $W(\theta, \psi)$ has measure zero by the Borel–Cantelli lemma.)

Rather than considering all decreasing functions ψ , one may consider the smaller class of *Khinchin sequences*: a function $\psi : \mathbb{N} \rightarrow (0, \infty)$ is called a Khinchin sequence if, in addition to the divergence condition $\sum_{q=1}^{\infty} \psi(q) = \infty$, the function $q \mapsto q\psi(q)$ is nonincreasing. Although less natural than the condition that ψ is decreasing, the hypothesis that a sequence is a Khinchin sequence is significant both for historical reasons (Khinchin first proved his eponymous theorem [3] in the setting of Khinchin sequences, although his theorem was later generalized) and because such sequences are often easier to work with.

Let θ be an irrational number and let ψ be a Khinchin sequence. A recent paper of D. H. Kim [5] gives a criterion, based on the continued fraction expansion of θ , for the set $W(\theta, \psi)$ to have full measure.¹ However, his paper leaves open the question of finding an analogue of Kurzweil's theorem in the setting of Khinchin sequences, although he proves several results in that direction [5, §3]. The aim of this paper is to complete the work of Kim by proving such an analogue.

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1. STATEMENT OF RESULTS

We first recall the main theorem of [5], rephrased slightly:²

Theorem 1.1 ([5, Theorem 2.1]). *Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and let $(q_k)_0^\infty$ be the sequence of the denominators of the convergents of θ . Let $\psi : \mathbb{N} \rightarrow (0, \infty)$ be a Khinchin sequence, and let $\phi(q) = 1/(q\psi(q))$. Then the following are equivalent:*

¹After this paper was written, Kim extended his result to all positive decreasing sequences in a joint paper with M. Fuchs [2].

²Technically, the result of [5] applies to the sets $\bigcap_{\varepsilon > 0} W(\theta, \varepsilon\psi)$ and not directly to the sets $W(\theta, \psi)$. But since the convergence or divergence of the series (1.1) is invariant under a slight perturbation of ψ , [5, Theorem 2.1] and Theorem 1.1 are equivalent.

- (A) $W(\theta, \psi)$ has full measure.
- (B) The series

$$(1.1) \quad \sum_{k=0}^{\infty} \frac{\log \phi(q_k) \wedge \log(q_{k+1}/q_k)}{\phi(q_k)}$$

diverges. (In this paper, \wedge and \vee denote minimum and maximum, respectively.)

To state our main theorem, we use the notation

$$\Sigma((a_i)_1^n : m)$$

to denote the sum of the m largest elements of the sequence $(a_i)_1^n$, with $\Sigma((a_i)_1^n : m) = \sum_1^m a_i$ if $m \geq n$. For $\alpha \geq 0$, we let $\Sigma((a_i)_1^n : \alpha) = \Sigma((a_i)_1^n : \lfloor \alpha \rfloor)$.

Theorem 1.2. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and let $(q_k)_0^\infty$ be the sequence of the denominators of the convergents of θ . Then the following are equivalent:

- (A) For every Khinchin sequence $\psi : \mathbb{N} \rightarrow (0, \infty)$, the set $W(\theta, \psi)$ has full measure.
- (B) For some $\varepsilon > 0$,

$$\limsup_{k \rightarrow \infty} \frac{1}{\log(q_k)} \Sigma \left(\left(\log \left(\frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k-1} : \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \right) < 1.$$

Remark. Since condition (B) of Theorem 1.2 is not equivalent to the condition that the sequence $(q_k)_1^\infty$, it follows from Kurzweil's theorem that condition (A) is not equivalent to the condition that for every decreasing positive function $\psi : \mathbb{N} \rightarrow (0, \infty)$ such that $\sum_{q=1}^\infty \psi(q) = \infty$, the set $W(\theta, \psi)$ has full measure. In particular, there exists a decreasing positive function $\psi : \mathbb{N} \rightarrow (0, \infty)$ such that $\sum_{q=1}^\infty \psi(q) = \infty$ and such that there is no Khinchin sequence $\psi' : \mathbb{N} \rightarrow (0, \infty)$ with $\psi'(q) \leq \psi(q)$ for all q . An example of such a sequence is given by the formula

$$\psi(q) = 2^{-n_k} \quad (2^{n_{k-1}} \leq q < 2^{n_k})$$

where $(n_k)_1^\infty$ is any sequence of integers such that $n_k - n_{k-1} \geq k$ for all k .

2. PROOF OF THEOREM 1.2

Convention. The symbol \asymp will denote a coarse multiplicative asymptotic, i.e. $A_n \asymp B_n$ means that there exists a constant $C > 0$ (the *implied constant*) such that $C^{-1}B_n \leq A_n \leq CB_n$.

Proof of (A) \Rightarrow (B). By contradiction, suppose that (B) is false. Then for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$\frac{1}{\log(q_{k_n})} \Sigma \left(\left(\log \left(\frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k_n-1} : \frac{1}{2^n} \frac{\log(q_{k_n})}{\log \log(q_{k_n})} \right) \geq 1 - \frac{1}{2^n}.$$

Without loss of generality, suppose that the sequence $(k_n)_1^\infty$ is increasing, and let $k_0 = 0$. For each $n \geq 1$, let S'_n be a subset of $\{0, \dots, k_n - 1\}$ of cardinality at most $\frac{1}{2^n} \frac{\log(q_{k_n})}{\log \log(q_{k_n})}$ such that

$$\sum_{k \in S'_n} \log \left(\frac{q_{k+1}}{q_k} \right) \geq (1 - 2^{-n}) \log(q_{k_n}).$$

Then let $S_n = S'_n \setminus \{0, \dots, k_{n-1} - 1\}$ and $T_n = \{k_{n-1}, \dots, k_n - 1\} \setminus S_n$. Then

$$(2.1) \quad \#(S_n) \leq \frac{1}{2^n} \frac{\log(q_{k_n})}{\log \log(q_{k_n})}$$

and

$$(2.2) \quad \sum_{k \in T_n} \log \left(\frac{q_{k+1}}{q_k} \right) \leq \log(q_{k_n}) - \sum_{k \in S'_n} \log \left(\frac{q_{k+1}}{q_k} \right) \leq 2^{-n} \log(q_{k_n}).$$

Now define the function $\phi : \mathbb{N} \rightarrow (0, \infty)$ by the formula

$$\phi(q) = \log(q_{k_n}) \quad \forall q_{k_{n-1}} \leq q < q_{k_n}.$$

Then ϕ is nondecreasing, and

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{1}{q\phi(q)} &= \sum_{n=1}^{\infty} \frac{1}{\log(q_{k_n})} \sum_{q=q_{k_{n-1}}}^{q_{k_n}-1} \frac{1}{q} \asymp \sum_{n=1}^{\infty} \frac{\log(q_{k_n}/q_{k_{n-1}})}{\log(q_{k_n})} \\ &= \sum_{n=1}^{\infty} \left[1 - \frac{\log(q_{k_{n-1}})}{\log(q_{k_n})} \right] \asymp \sum_{n=1}^{\infty} 1 \wedge \log \left(\frac{\log(q_{k_n})}{\log(q_{k_{n-1}})} \right) = \infty. \end{aligned}$$

Thus $\psi(q) = 1/(q\phi(q))$ is a Khinchin sequence. So by (A) together with Theorem 1.1, the series (1.1) diverges. On the contrary, we show that (1.1) converges:

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{\log \phi(q_k) \wedge \log(q_{k+1}/q_k)}{\phi(q_k)} \\ &\leq \sum_{n=1}^{\infty} \left[\sum_{k \in S_n} \frac{\log \phi(q_k)}{\phi(q_k)} + \sum_{k \in T_n} \frac{\log(q_{k+1}/q_k)}{\phi(q_k)} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{\log \log(q_{k_n})}{\log(q_{k_n})} \#(S_n) + \frac{1}{\log(q_{k_n})} \sum_{k \in T_n} \log(q_{k+1}/q_k) \right] \\ &\leq \sum_{n=1}^{\infty} \left[\frac{1}{2^n} + \frac{1}{2^n} \right] \quad (\text{by (2.1) and (2.2)}) \\ &= 2 < \infty. \end{aligned}$$

This contradiction completes the proof. \square

Proof of (B) \Rightarrow (A). Let $\psi : \mathbb{N} \rightarrow \infty$ be a Khinchin sequence, and by contradiction suppose that $W(\theta, \psi)$ does not have full measure. Then by Theorem 1.1, the series (1.1) converges, where $\phi(q) = 1/(q\psi(q))$ is nondecreasing. Let

$$S = \{k : \phi(q_k) \leq q_{k+1}/q_k\}, \quad T = \mathbb{N} \setminus S,$$

so that

$$\infty > \sum_{k=0}^{\infty} \frac{\log \phi(q_k) \wedge \log(q_{k+1}/q_k)}{\phi(q_k)} = \sum_{k \in S} \frac{\log \phi(q_k)}{\phi(q_k)} + \sum_{k \in T} \frac{\log(q_{k+1}/q_k)}{\phi(q_k)}.$$

For each $m \in \mathbb{N}$, let Q_m be the largest integer such that $\phi(Q_m) \leq 2^m$. Then

$$\begin{aligned} \frac{1}{\phi(q)} &\asymp \sum_{m \in \mathbb{N}: 2^m \geq \phi(q)} \frac{1}{2^m} = \sum_{m \in \mathbb{N}: q \leq Q_m} \frac{1}{2^m} \\ \frac{\log \phi(q)}{\phi(q)} &\asymp \sum_{m \in \mathbb{N}: 2^m \geq \phi(q)} \frac{m}{2^m} = \sum_{m \in \mathbb{N}: q \leq Q_m} \frac{m}{2^m} \end{aligned}$$

and thus

$$\begin{aligned} \infty &= \sum_{q=1}^{\infty} \frac{1}{q\phi(q)} \asymp \sum_{m=0}^{\infty} \sum_{q=1}^{Q_m} \frac{1}{q2^m} \asymp \sum_{m=0}^{\infty} \frac{\log(Q_m)}{2^m} \\ \infty &> \sum_{k \in S} \frac{\log \phi(q_k)}{\phi(q_k)} + \sum_{k \in T} \frac{\log(q_{k+1}/q_k)}{\phi(q_k)} \\ &\asymp \sum_{m=0}^{\infty} \left[\frac{m}{2^m} \# \{k \in S : q_k \leq Q_m\} + \frac{1}{2^m} \sum_{\substack{k \in T \\ q_k \leq Q_m}} \log \left(\frac{q_{k+1}}{q_k} \right) \right]. \end{aligned}$$

It follows that if

$$\begin{aligned} \lambda_m &= \frac{\frac{m}{2^m} \#\{k \in S : q_k \leq Q_m\} + \frac{1}{2^m} \sum_{\substack{k \in T \\ q_k \leq Q_m}} \log\left(\frac{q_{k+1}}{q_k}\right)}{\frac{\log(Q_m)}{2^m}} \\ &= \frac{1}{\log(Q_m)} \left[m \#\{k \in S : q_k \leq Q_m\} + \sum_{\substack{k \in T \\ q_k \leq Q_m}} \log\left(\frac{q_{k+1}}{q_k}\right) \right], \end{aligned}$$

then

$$\liminf_{m \rightarrow \infty} \lambda_m = 0.$$

On the other hand, if

$$\kappa_m = \frac{m}{2^m} \#\{k \in S : q_k \leq Q_m\},$$

then

$$\lim_{m \rightarrow \infty} \kappa_m = 0.$$

Fix $\varepsilon > 0$, and choose $m \geq 2$ such that $\lambda_m, \kappa_m \leq \varepsilon$. Then

$$\frac{m}{2^m} \vee \frac{m}{\log(Q_m)} \leq \frac{\varepsilon}{\#\{k \in S : q_k \leq Q_m\}}.$$

Consider the function

$$f(x) = \frac{x}{2^x} \vee \frac{x}{\log(Q_m)} \quad (x \geq 2).$$

Since f is the maximum of an increasing function and a decreasing function, f has a unique minimum, which occurs when the two inputs to the maximum agree, namely at $x = \log_2 \log(Q_m)$. Thus

$$\frac{\varepsilon}{\#\{k \in S : q_k \leq Q_m\}} \geq f(m) \geq \min(f) = \frac{\log_2 \log(Q_m)}{\log(Q_m)} \geq \frac{\log \log(Q_m)}{\log(Q_m)}$$

i.e.

$$\#\{k \in S : q_k \leq Q_m\} \leq \frac{\varepsilon \log(Q_m)}{\log \log(Q_m)}.$$

On the other hand, since $\lambda_m \leq \varepsilon$,

$$\sum_{\substack{k \in T \\ q_k \leq Q_m}} \log\left(\frac{q_{k+1}}{q_k}\right) \leq \varepsilon \log(Q_m).$$

Let k_m be the smallest integer such that $Q_m < q_{k_m}$. Then

$$\begin{aligned} \#\{k \in S : k < k_m\} &\leq \frac{\varepsilon \log(q_{k_m})}{\log \log(q_{k_m})} \\ \sum_{\substack{k \in T \\ k < k_m}} \log\left(\frac{q_{k+1}}{q_k}\right) &\leq \varepsilon \log(q_{k_m}) \end{aligned}$$

and thus

$$\Sigma \left(\left(\log\left(\frac{q_{i+1}}{q_i}\right) \right)_{i=0}^{k_m-1} : \frac{\varepsilon \log(q_{k_m})}{\log \log(q_{k_m})} \right) \geq (1 - \varepsilon) \log(q_{k_m}).$$

Since ε was arbitrary and $k_m \rightarrow \infty$, for all $\varepsilon > 0$ we have

$$\limsup_{k \rightarrow \infty} \frac{1}{\log(q_k)} \Sigma \left(\left(\log\left(\frac{q_{i+1}}{q_i}\right) \right)_{i=0}^{k-1} : \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \right) = 1,$$

contradicting (B). □

3. CONSEQUENCES OF THEOREM 1.2

In this section we use Theorem 1.2 to prove some necessary and sufficient conditions on θ for $W(\theta, \psi)$ to be full measure for every Khinchin sequence ψ , including reproving some results from [5, §3]. For convenience let

$$\Omega = \{\theta \in \mathbb{R} : \text{for every Khinchin sequence } \psi, \text{ the set } W(\theta, \psi) \text{ has full measure}\}.$$

In other words, Ω is the set of all θ such that the equivalent conditions of Theorem 1.2 hold.

Theorem 3.1. *Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and let $(q_k)_0^\infty$ be the sequence of the denominators of the convergents of θ .*

(i) *If*

$$(3.1) \quad \limsup_{k \rightarrow \infty} \frac{\log(q_k)}{k} < \infty,$$

then $\theta \in \Omega$.

(ii) *If*

$$(3.2) \quad \limsup_{k \rightarrow \infty} \frac{\log(q_k)}{k \log(k)} = \infty,$$

then $\theta \notin \Omega$.

(iii) *If*

$$(3.3) \quad \sum_{k=2}^{\infty} \frac{1}{\log(q_k)} < \infty$$

then $\theta \notin \Omega$.

(iv) *If*

$$(3.4) \quad \limsup_{k \rightarrow \infty} \frac{q_{k+1}/q_k}{\log(q_k)} < \infty,$$

then $\theta \in \Omega$.

(v) *If*

$$(3.5) \quad \limsup_{k \rightarrow \infty} \frac{\log(q_{k+1}/q_k)}{\log(q_k)} = \infty,$$

then $\theta \notin \Omega$.

Remark. Parts (i), (iii), and (iv) correspond to [5, Theorem 3.1 and Proposition 3.2]. Although in some cases the new proofs are not shorter than the old proofs, having two proofs may bring further insight.

Remark. By well-known facts about continued fractions (e.g. [4, Theorems 9 and 13]), the conditions (3.4) and (3.5) have interpretations in terms of Diophantine approximation:

- θ satisfies (3.4) if and only if for some $\varepsilon > 0$, θ is not ψ -approximable, where

$$\psi(q) = \frac{\varepsilon}{q^2 \log(q)}.$$

We recall that a number θ is called ψ -approximable if there exist infinitely many rationals $p/q \in \mathbb{Q}$ such that

$$\left| \theta - \frac{p}{q} \right| < \psi(q).$$

- θ satisfies (3.5) if and only if θ is a Liouville number. We recall that a number θ is called *Liouville* if for all $n \in \mathbb{N}$, θ is ψ_n -approximable, where $\psi_n(q) = q^{-n}$.

Remark. Any badly approximable number θ satisfies both (3.1) and (3.4), so $\text{BA} \subseteq \Omega$. This can also be seen from Kurzweil's theorem.

Remark. The continued fraction expansion of e (see e.g. [1]) satisfies (3.4), so $e \in \Omega$.

Proof of (i). Choose $M < \infty$ so that for all k , $\log(q_k) \leq Mk$. Let $\varepsilon > 0$ be arbitrary (e.g. $\varepsilon = 1$). Then for sufficiently large k ,

$$\frac{\varepsilon \log(q_k)}{\log \log(q_k)} \leq \frac{\varepsilon Mk}{\log(Mk)} \leq \frac{k}{8}.$$

Let $S \subseteq \{0, \dots, k-1\}$ be a subset of cardinality at most $k/8$, and let $T = \{0, \dots, k-1\} \setminus S$. A counting argument shows that

$$\#\{i = 0, \dots, k-1 \text{ even} : i, i+1 \in T\} \geq k/4,$$

and thus

$$\sum_{i \in T} \log(q_{i+1}/q_i) \geq \sum_{\substack{i \text{ even} \\ i, i+1 \in T}} \log(q_{i+2}/q_i) \geq (k/4) \log(2) \geq \frac{\log(2)}{4M} \log(q_k).$$

It follows that

$$\frac{1}{\log(q_k)} \Sigma \left(\left(\log \left(\frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k-1} : \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \right) \leq 1 - \frac{\log(2)}{4M}.$$

To complete the proof, we take the limsup as $k \rightarrow \infty$ and then apply Theorem 1.2. \square

Proof of (ii). Fix $\varepsilon > 0$. By assumption, there exist infinitely many k satisfying

$$\log(q_k) \geq \frac{2}{\varepsilon} k \log(k).$$

For such k ,

$$\frac{\varepsilon \log(q_k)}{\log \log(q_k)} \geq \frac{\varepsilon(2/\varepsilon)k \log(k)}{\log((2/\varepsilon)k \log(k))} \geq \frac{2k \log(k)}{\log(k^2)} = k,$$

where the middle inequality holds for all k sufficiently large. But then

$$\Sigma \left(\left(\log \left(\frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k-1} : \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \right) = \sum_{i=0}^{k-1} \log \left(\frac{q_{i+1}}{q_i} \right) = \log(q_k).$$

To complete the proof, we divide by $\log(q_k)$, take the limsup as $k \rightarrow \infty$, and apply Theorem 1.2. \square

Since (3.3) implies (3.2), (iii) does not require a separate proof.

Proof of (iv). Choose $M < \infty$ such that for all k , $q_{k+1}/q_k \leq M \log(q_k)$. Then for all $\varepsilon > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned} \Sigma \left(\left(\log \left(\frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k-1} : \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \right) &\leq \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \max \left\{ \log \left(\frac{q_{i+1}}{q_i} \right) : i = 0, \dots, k-1 \right\} \\ &\leq \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \log(M \log(q_k)) \leq 2\varepsilon \log(q_k), \end{aligned}$$

where the last inequality holds for all k large enough such that $q_k \geq e^M$. To complete the proof, we let $\varepsilon = 1/4$, divide by $\log(q_k)$, take the limsup as $k \rightarrow \infty$, and apply Theorem 1.2. \square

Proof of (v). The assumption (3.5) implies that

$$\limsup_{k \rightarrow \infty} \frac{\log(q_k/q_{k-1})}{\log(q_k)} = 1.$$

Fix $\varepsilon > 0$. By assumption, there exist infinitely many k such that

$$\log(q_k/q_{k-1}) \geq (1 - \varepsilon) \log(q_k).$$

For such k , if we assume that k is chosen large enough so that $\frac{\varepsilon \log(q_k)}{\log \log(q_k)} \geq 1$, then

$$\Sigma \left(\left(\log \left(\frac{q_{i+1}}{q_i} \right) \right)_{i=0}^{k-1} : \frac{\varepsilon \log(q_k)}{\log \log(q_k)} \right) \geq \log \left(\frac{q_k}{q_{k-1}} \right) \geq (1 - \varepsilon) \log(q_k).$$

To complete the proof, we divide by $\log(q_k)$, take the limsup as $k \rightarrow \infty$, use the fact that ε was arbitrary, and apply Theorem 1.2. \square

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